Linear Algebra II

07/04/2014, Monday, 9:00-12:00

You are **NOT** allowed to use any type of calculators.

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1 (15 pts) **Gram-Schmidt process**

Consider the vector space P_2 with the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x.$$

Apply the Gram-Schmidt process to transform the basis $\{1, x, x^2\}$ into an orthonormal basis.

REQUIRED KNOWLEDGE: **inner product, Gram-Schmidt process**

SOLUTION:

To apply the Gram-Schmidt process, we first note that

$$\begin{aligned}
\langle 1, 1 \rangle &= \int_0^1 1 \cdot 1\,\mathrm{d}x = x|_0^1 = 1 \\
\langle 1, x \rangle &= \int_0^1 1 \cdot x\,\mathrm{d}x = \frac{x^2}{2}|_0^1 = \frac{1}{2} \\
\langle 1, x^2 \rangle &= \int_0^1 1 \cdot x^2\,\mathrm{d}x = \frac{x^3}{3}|_0^1 = \frac{1}{3} \\
\langle x, x \rangle &= \int_0^1 x \cdot x\,\mathrm{d}x = \frac{x^3}{3}|_0^1 = \frac{1}{3} \\
\langle x, x^2 \rangle &= \int_0^1 x \cdot x^2\,\mathrm{d}x = \frac{x^4}{4}|_0^1 = \frac{1}{4} \\
\langle x^2, x^2 \rangle &= \int_0^1 x^2 \cdot x^2\,\mathrm{d}x = \frac{x^5}{5}|_0^1 = \frac{1}{5}.
\end{aligned}$$

By applying the Gram-Schmidt process, we obtain:

$$u_1 = \frac{1}{\|1\|}$$

$$u_1 = 1$$

$$u_2 = \frac{x - p_1}{\|x - p_1\|}$$

$$\begin{aligned} p_1 &= \langle x, 1\rangle \cdot 1 \\ &= \frac{1}{2} \\ x - p_1 &= x - \frac{1}{2} \\ \|x - p_1\|^2 &= \langle x - \frac{1}{2}, x - \frac{1}{2}\rangle = \langle x, x\rangle - 2 \cdot \frac{1}{2}\langle x, 1\rangle + \frac{1}{4}\langle 1, 1\rangle \\ \|x - p_1\|^2 &= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12} \\ \|x - p_1\| &= \frac{1}{2\sqrt{3}} \end{aligned}$$

$$u_2 = 2\sqrt{3}(x - \frac{1}{2})$$

$$u_3 = \frac{x^2 - p_2}{\|x^2 - p_2\|}$$

$$\begin{aligned} p_2 &= \langle x^2, 1\rangle \cdot 1 + \langle x^2, 2\sqrt{3}(x - \frac{1}{2})\rangle \cdot 2\sqrt{3}(x - \frac{1}{2}) \\ &= \langle x^2, 1\rangle + 12\langle x^2, x - \frac{1}{2}\rangle(x - \frac{1}{2}) \\ &= \langle x^2, 1\rangle + 12\big(\langle x^2, x\rangle - \frac{1}{2}\langle x^2, 1\rangle\big)(x - \frac{1}{2}) \\ &= \frac{1}{3} + 12 \cdot (\frac{1}{4} - \frac{1}{6})(x - \frac{1}{2}) = \frac{1}{3} + (x - \frac{1}{2}) = x - \frac{1}{6} \\ x^2 - p_2 &= x^2 - x + \frac{1}{6} \\ \|x^2 - p_2\|^2 &= \langle x^2, x^2\rangle + \langle x, x\rangle + \frac{1}{36}\langle 1, 1\rangle - 2\langle x^2, x\rangle + 2 \cdot \frac{1}{6}\langle x^2, 1\rangle - 2 \cdot \frac{1}{6}\langle 1, x\rangle \\ \|x^2 - p_2\|^2 &= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - 2 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{180} \\ \|x^2 - p_2\| &= \frac{1}{6\sqrt{5}} \end{aligned}$$

$$u_3 = 6\sqrt{5}(x^2 - x + \frac{1}{6}).$$

(a) Consider the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Find real numbers a and b such that $(M - aI)(M - bI) = 0$.

(b) Consider the matrix

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Find real numbers a and b such that $(M - aI)(M - bI) = 0$.

REQUIRED KNOWLEDGE: **Cayley-Hamilton theorem**

SOLUTION:

(2a): The characteristic polynomial of M can be found as

$$\det(\lambda I - M) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5).$$

Cayley-Hamilton theorem implies that

$$(M + I)(M - 5I) = 0.$$

As such, one can take $a = -1$ and $b = 5$.

(2b): For this case, the characteristic polynomial of M can be found as

$$\det(\lambda I - M) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = (\lambda - 1 - i)(\lambda - 1 + i).$$

Suppose that a and b are real numbers satisfying

$$(M - aI)(M - bI) = M^2 - (a+b)M + abI = 0.$$

From Cayley-Hamilton theorem, we know that

$$M^2 - 2M + 2I = 0.$$

Subtracting the last two, we get

$$(2 - a - b)M + (ab - 2)I = 0.$$

Since M and I are linearly independent, we have

$$a + b = 2 \quad \text{and} \quad ab = 2.$$

In other words, a and b are the roots of the polynomial $x^2 - 2x + 2$. Since this polynomial has only complex roots $(1 \pm i)$, there are no real values a and b such that $(M - aI)(M - bI) = 0$.

Consider the matrix

$$M = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}.$$

(a) Show that the singular values of M are $\sigma_1 = 5$ and $\sigma_2 = 3$.

(b) Find a singular value decomposition for M.

(c) Find the best rank 1 approximation of M.

REQUIRED KNOWLEDGE: **singular value decomposition, lower rank approximations.**

SOLUTION:

(3a):

Note that

$$M^TM = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.$$

Then, the characteristic polynomial of M^TM can be found as

$$\begin{aligned} p_{M^TM}(\lambda) &= \det\left(\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix}\right) \\ &= (\lambda-17)^2 - 8^2 = (\lambda-25)(\lambda-9). \end{aligned}$$

Then, the eigenvalues of M^TM are given by

$$\lambda_1 = 25 \quad \text{and} \quad \lambda_2 = 9$$

and hence the singular values by

$$\sigma_1 = 5 \quad \text{and} \quad \sigma_2 = 3.$$

(3b):

Next, we need to diagonalize M^TM in order to obtain the orthogonal matrix V. To do so, we first compute eigenvectors of M^TM.

For the eigenvalue $\lambda_1 = 25$, we have

$$0 = \left(M^TM - 25I\right)x = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This results in the following eigenvector

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = 9$, we have

$$0 = \left(M^TM - 9I\right)x = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This results, for instance, in the following eigenvector

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence, we get

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Note that the rank of M is equal to the number of nonzero singular values. Thus, $r = \text{rank}(M) = 2$. By using the formula

$$u_i = \frac{1}{\sigma_i} M v_i$$

for $i = 1, 2$, we obtain

$$u_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{3}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$

The last column vector of the matrix U can be found by looking at the null space of M^T:

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0.$$

This yields

$$u_3 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

Finally, the SVD can be given by:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(3c):

The best rank 1 approximation can be obtained as follows:

$$\tilde{M} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{2}} & 0 \\ \frac{5}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \\ 0 & 0 \end{bmatrix}.$$

Let A be an $n \times n$ matrix.

(a) Let λ be a real eigenvalue of A. Show that

$$\sigma_n \leqslant |\lambda| \leqslant \sigma_1$$

where σ_1 and σ_n are the largest and the smallest singular values of A, respectively.

(b) Suppose that A is symmetric. Show that $|\lambda|$ is a singular value of A if λ is an eigenvalue of A.

REQUIRED KNOWLEDGE: **eigenvalues and singular values.**

SOLUTION:

(4a):

Let (λ, x) be an eigenpair of A, that is

$$Ax = \lambda x.$$

Let $A = U\Sigma V^T$ be a singular value decomposition of A. Note that

$$\|U\Sigma V^T x\|_F = |\lambda| \, \|x\|_F.$$

Note that if Q is an orthogonal matrix $\|Qv\|_F = \|v\|_F$ for all v. Then, we have

$$\|U\Sigma V^T x\|_F = \|\Sigma V^T x\|_F \quad \text{and} \quad \|x\|_F = \|V^T x\|_F.$$

Let $y = V^T x$. Then, we get

$$\|\Sigma y\|_F = |\lambda| \, \|y\|_F.$$

Further, let $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$. Then, we have

$$|\lambda|^2 = \frac{\sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 + \cdots + \sigma_n^2 y_n^2}{y_1^2 + y_2^2 + \cdots + y_n^2}.$$

Since $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n$, we get

$$\frac{\sigma_1^2 y_1^2 + \sigma_1^2 y_2^2 + \cdots + \sigma_1^2 y_n^2}{y_1^2 + y_2^2 + \cdots + y_n^2} \geqslant |\lambda|^2 \geqslant \frac{\sigma_n^2 y_1^2 + \sigma_n^2 y_2^2 + \cdots + \sigma_n^2 y_n^2}{y_1^2 + y_2^2 + \cdots + y_n^2}.$$

Consequently, we obtain

$$\sigma_1 \geqslant |\lambda| \geqslant \sigma_2.$$

(4b):

Note that if λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2. Since A is symmetric, we know that $A^2 = A^T A$ and its eigenvalues are real. Therefore, $|\lambda|$ is a singular value of A.

(a) Consider the function

$$f(x, y) = x^3 + y^3 - 3xy.$$

Find the stationary points of f and determine whether its stationary points are local minimum/maximum or saddle points.

(b) Let

$$M = \begin{bmatrix} a & -a & 0 \\ -a & b & a \\ 0 & a & a \end{bmatrix}$$

where a and b are real numbers. Determine all values of a and b for which M is positive definite.

REQUIRED KNOWLEDGE: **stationary points, positive definiteness.**

SOLUTION:

(5a): In order to find the stationary points, we need the partial derivatives:

$$f_x = 3x^2 - 3y \quad \text{and} \quad f_y = 3y^2 - 3x.$$

Then, $(\bar{x}, \bar{y})$ is a stationary point if and only if

$$3\bar{x}^2 - 3\bar{y} = 0$$
$$3\bar{y}^2 - 3\bar{x} = 0.$$

This leads to $\bar{x}^4 = \bar{x}$, or equivalently $\bar{x}(\bar{x}^3 - 1) = 0$. Hence, we have $\bar{x} = 0$ or $\bar{x} = 1$ since $\bar{x}^3 - 1 = (\bar{x} - 1)(\bar{x}^2 + \bar{x} + 1)$. Then, the stationary points are $(\bar{x}, \bar{y}) = (0, 0)$ or $(\bar{x}, \bar{y}) = (1, 1)$. To determine the character of these points, we need the second order partial derivatives:

$$f_{xx} = 6x, \quad f_{xy} = -3, \quad \text{and} \quad f_{yy} = 6y.$$

For the stationary point $(\bar{x}, \bar{y}) = (0, 0)$, we have

$$H_{(0,0)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}.$$

Note that the characteristic polynomial of the Hessian matrix $H_{(0,0)}$ is given by $\lambda^2 - 9$. Hence, it has one positive ($\lambda_1 = 3$) and one negative eigenvalue ($\lambda_2 = -3$). Therefore, $H_{(0,0)}$ is indefinite and the stationary point $(\bar{x}, \bar{y}) = (0, 0)$ is a saddle point.

For the stationary point $(\bar{x}, \bar{y}) = (1, 1)$, we have

$$H_{(1,1)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(1,1)} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

Note that the characteristic polynomial of the Hessian matrix $H_{(1,1)}$ is given by $(\lambda - 6)^2 - 9$. Hence, it has two positive eigenvalues $\lambda_1 = 9$ and $\lambda_2 = 3$. This means that $H_{(1,1)}$ is positive definite. Consequently, stationary point $(\bar{x}, \bar{y}) = (1, 1)$ corresponds to a local minimum.

(5b): A symmetric matrix is positive definite if and only if all its leading principal minors are positive. Note that the leading principal minors of M are given by:

$$\det(a), \quad \det\left(\begin{bmatrix} a & -a \\ -a & b \end{bmatrix}\right) = ab - a^2, \quad \text{and} \quad \det\left(\begin{bmatrix} a & -a & 0 \\ -a & b & a \\ 0 & a & a \end{bmatrix}\right) = a(ab - a^2) - a^3 = a^2b - 2a^3.$$

Then, the matrix M is positive definite if and only if

$$a > 0, \quad ab - a^2 > 0, \quad \text{and} \quad a^2 b - 2a^3 > 0.$$

This is, however, equivalent to saying that

$$a > 0, \quad b - a > 0, \quad \text{and} \quad b - 2a > 0.$$

Note that the inequality in the middle is implied by the last one. Hence, we can conclude that the matrix M is positive definite if and only if

$$a > 0 \quad \text{and} \quad b > 2a.$$

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Is A diagonalizable? Why?

(c) Put A into the Jordan canonical form.

REQUIRED KNOWLEDGE: **eigenvalues/vectors, Jordan canonical form, diagonalization.**

SOLUTION:

(6a): Note that A is upper triangular. As such, the eigenvalues can be read from the diagonal $\lambda_{1,2,3,4} = 1$.

(6b): It is diagonalizable if and only if it has 4 linearly independent eigenvectors. To find the eigenvectors, we need to solve the equation $(A - I)x = 0$:

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

This leads to $x_3 = x_4 = 0$. Thus, eigenvectors of A must be of the form

$$x = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

This means that we can find at most two linearly independent eigenvectors. Therefore, A is not diagonalizable.

(6c): Since there are at most two linearly independent eigenvectors, Jordan canonical form consists of two blocks of sizes $3 + 1$ or $2 + 2$. Note that

$$(A - I)^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A - I)^3 = 0.$$

Next, we check if

$$(A - I)^2 v = x$$

has a solution where x is an eigenvector. Note that

$$(A - I)^2 v = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_4 \\ v_4 \\ 0 \\ 0 \end{bmatrix} = v_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, we have

$$(A - I)^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

This means that

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, (A-I) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, (A-I)^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

would generate a cyclic subspace. Thus, we can conclude that Jordan canonical form consists of two blocks of sizes $3+1$. Finally, we need to choose an eigenvector that is not linearly dependent to $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$. This is achieved, for instance, by the eigenvector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, we have

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{J}.$$